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ENTROPY AND KNOTS[1]

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ABSTRACT. We show that a smooth flow on S^3 with positive topological entropy must possess periodic closed orbits in infinitely many different knot type equivalence classes.

We are concerned here with smooth flows ϕ_t (= autonomous ordinary differential equations = vector fields) on real three-dimensional space $\mathbf{R}^3$ or its compactification S^3. Periodic orbits form perhaps the most obvious geometric aspect of these flows. An orbit is periodic provided, for some value of t, say T, $\phi_T(x_0) = x_0$ for some (and hence any) x_0 on the orbit. It follows that $\phi_{t+T}(x_0) = \phi_t(x_0)$ for all t, and this *trajectory* of x_0 is a smoothly embedded circle or "one-sphere" S^1. To a topologist, such an embedded S^1 is a knot—which might be the trivial knot (or unknot).

Other research has concerned itself with what kinds of knots occur and has shown that infinitely many kind of (inequivalent) knots occur under certain geometric hypotheses [**M**, **B-W I, II**]. Our work was motivated by trying to get such a conclusion with a minimal hypothesis, perhaps nongeometric. Our main result is

THEOREM. *If ϕ_t is a C^r flow on $\mathbf{R}^3$ or S^3 such that either*
(a) $r \geqslant 1$ *and ϕ_t has a hyperbolic periodic orbit with a transverse homoclinic point, or*
(b) $r \geqslant 2$ *and ϕ_t has a compact invariant set with positive topological entropy,*
then among the closed orbits there are infinitely many distinct knot types.

We need only prove this result using hypothesis (a), as A. Katok [**K**] has shown that (b) implies (a).

(While Katok's article is written in the setting of mappings rather than flows, he informs us that the result is equally valid in the flow case which we use here.)

We were in the process of trying to prove this theorem when we first learned of Bennequin's theorem (see (1.2) below) which estimates the *genus* of a knot presented as a braid. This beautiful theorem is the main tool to prove the existence of infinitely many knot types among the closed orbits of flows satisfying the hypothesis of our theorem.

The paper is organized as follows: Basic definitions are given in §1 along with a formulation of Bennequin's theorem. The second section concerns symbolic dynamics and "Templates", a word we feel is more appropriate than "knotholders"

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discussed earlier in [**B-W I, II**]. Next we show how Alexander's trick, of isotoping *any* knot to a *braid*, can be used to isotope a template to a "braided template". This allows one to give an infinite family of knots K_n with the genus of K_n quadratic in n and hence $\to \infty$ with n. There are in fact three cases depending upon a two-by-two matrix which describes how the two ribbons of a certain braided template cross one another.

1. Basic definitions and theorems of Bennequin and Katok. Given a smooth flow ϕ_t on a manifold M of dimension n, a periodic orbit $\mathcal{O}$ of ϕ_t is *hyperbolic* provided the tangent bundle of M^n restricted to $\mathcal{O}$ has two subbundles E^u, E^s, invariant under $d\phi$ such that

(a) $TM^n|\mathcal{O} = E^u \oplus E^s \oplus T'$ where T' is tangent to the flow along $\mathcal{O}$;
(b) $|d\phi_t(v)| \geqslant ce^{\lambda|v|}$ for $v \in E^u$, and
(c) $|d\phi_t(v)| \leqslant ce^{-\lambda|v|}$, $v \in E^s$ where $c > 0$ and $\lambda > 1$.

In the setting of this paper, each of the bundles E^s, E^u and T' are one-dimensional and in this case $\mathcal{O}$ has a two-dimensional *stable manifold* W^s tangent to $E^s \oplus T'$ along $\mathcal{O}$ and two-dimensional *unstable manifold* W^u tangent to $E^u \oplus T'$ along $\mathcal{O}$. The manifold W^s (respectively W^u) is filled by orbits of ϕ_t which are asymptotic to $\mathcal{O}$ in positive (respectively negative) time. One says $\mathcal{O}$ has a transverse homoclinic point provided $W^u(\mathcal{O})$ and $W^s(\mathcal{O})$ have a point of transverse intersection, which is not on $\mathcal{O}$. The full intersection will consist of orbit(s) which are asymptotic to $\mathcal{O}$ in both positive and negative time.

(1.1) KATOK'S THEOREM [**K**]. *A* $C^{1+\alpha}$ $(\alpha > 0)$ *flow* ϕ_t *on a three-dimensional manifold, with positive topological entropy, has a hyperbolic closed orbit with a transverse homoclinic point.*

The reader is referred elsewhere, [**A-K-M**, **G-H**, §5.8] for a definition of entropy.

By a *knot* is meant a smoothly embedded one-sphere S^1 in $\mathbf{R}^3$ or S^3. It is convenient to have a chosen *orientation* on S^1. Given a knot K, there is a smooth oriented surface M^2 embedded in $\mathbf{R}^3$ or S^3 such that K is the boundary of M^2. Among all such surfaces, the minimal genus g is called the *genus* of K. The genus of

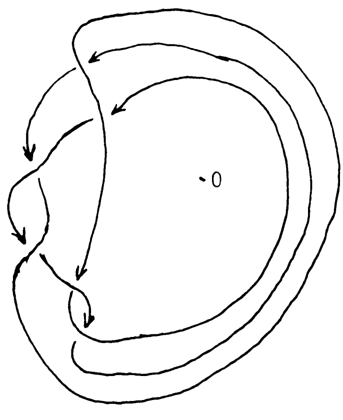
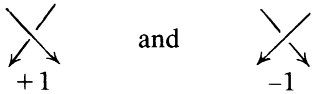


FIGURE (1.1). A braid on 3 strands

a knot is by definition a knot invariant, though it is not easily seen to be so important as it really is. If a knot is arranged so that its projection on some plane passes in the same (say counterclockwise) direction about the origin of the plane, we say it is a *braid* (see Figure (1.1)).

If this placement is generic, there will be only finitely many double points. All rays from the origin that miss the double points will interest the projected knot in the same number of points, called the number of *strands* in the braid. The crossings occur in two types

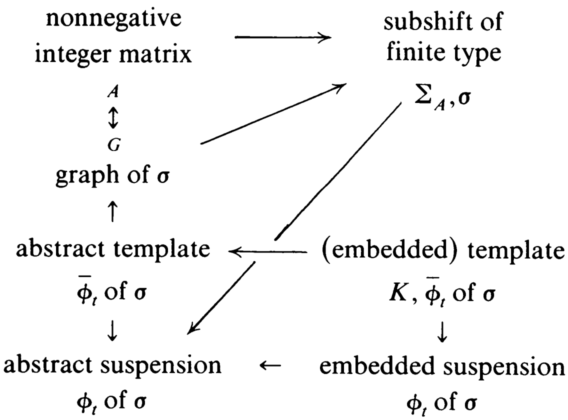


which we call *positive* and *negative*, though this choice is by no means universal. We can now add up the crossing numbers over a braid and obtain the *algebraic crossing number*.

(1.2) THEOREM (BENNEQUIN [**B**]). *If a knot K is arranged as a braid, then $2g(K) > |c| - n$ where $g(K)$ is the genus of K, c is the algebraic crossing number and n is the number of strands of the braid.*

2. Symbolic dynamics and templates. In this section we discuss several symbolic and geometric devices, which present a qualitative description of part of a smooth flow. Before making formal definitions and describing these objects we present a chart showing their relationships.

Arrows indicate that we can pass from one to another, perhaps with loss of information.



Let A be a square $n \times n$ matrix whose entries are nonnegative integers. The corresponding *graph* consists of n vertices $v_1, \ldots, v_n$ with A_{ij} arrows (oriented edges) passing from the ith to the jth vertex, for each i, j.

Let E be the finite set of oriented edges of this graph. Let E^z denote the space of all doubly infinite sequences $\{x_i\}_{i=-\infty}^{\infty}$ of elements of E. We define a closed subspace $\Sigma_A \subset E^z$ by stipulating that the sequence $\underline{x} = (\ldots, x_1, x_0, x_1, \ldots)$ is in Σ_A if and only if for each $i \in Z$ the edge x_i ends at the vertex of the graph where edge x_{i+1} begins. Σ_A is endowed with a topology in which the coordinates x_i with $|i|$ small are important, e.g., the topology induced from the product topology on $E^{\mathbb{Z}}$. Equivalently, the metric on Σ_A given by

$$d(\bar{x}, \bar{y}) = \sum_{i=-\infty}^{\infty} \Delta(x_i, y_i) \cdot 2^{-|i|}$$

where $\Delta(j, k) = 1$ if $j \neq k$ and $\Delta(j, j) = 0$. The shift map $\sigma: \Sigma_A \to \Sigma_A$ merely moves the coordinates over one. There is the usual problem as to which way—and this is tied up with the problem of which side one composes matrices on. Symbolic dynamicists most often shift to the left and define $\sigma(\underline{x})_i = x_{i+1}$.

(2.1) DEFINITION. The compact metric space Σ_A, together with the shift homeomorphism $\sigma: \Sigma_A \to \Sigma_A$ is called the *subshift of finite type* corresponding to the matrix A.

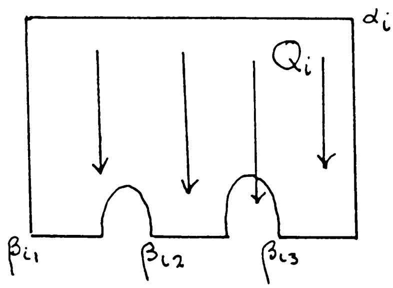


FIGURE (2.1)

We now wish to formally describe the concept of an abstract template. We start with a graph G constructed as above from an n-by-n nonnegative integer matrix A. To each vertex of G we associated a region Q in the plane as shown in Figure (2.1).

We have illustrated it with 3 strips extending to the bottom, but in general for vertex i, we construct a Q_i with $\Sigma_j A_{ij}$ strips at the bottom. On Q_i we construct a (partial) flow which is given by the unit downward vector field. Thus this flow enters on the top interval of Q_i (which we call α_i) and exits on the $\Sigma_j A_{ij}$ bottom intervals (called $\{\beta_{ik}\}$) together with the "arch intervals" between them. The two sides of Q_i are tangent to the flow.

To form a template T from this data, for each i and j we attach A_{ij} of the intervals $\{\beta_{ik}\}$ to α_j forming a branched surface (see Figure (2.2)).

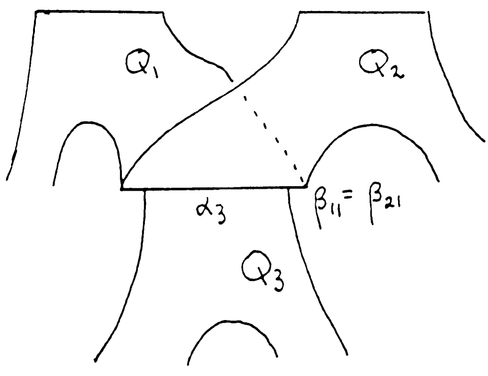


FIGURE (2.2)

We do this in such a way that the attaching map is overflowing and expanding, i.e., we require that α_j is in the interior of β_{ik} and that the attaching map from an interior subinterval of β_{ik} to α_j strictly expands lengths. If we patch together the vector fields on the Q_i's we obtain a (partial) semiflow $\bar{\phi}_t$ on the "branched surface" $T = UQ_i$. Note that the semiflow $\bar{\phi}_t$: $T \to T$ fails to be a flow in two ways: It is well defined only for positive time on the branch intervals $\{\alpha_j\}$, and secondly some of the flow lines leave T (through "arches" and parts of β's which are not attached to any α).

(2.2) DEFINITION. A branched surface T with semiflow $\bar{\phi}_t$ constructed in this way is called a *template* associated with the matrix A.

A template T is *orientable* if it is constructed in such a way that it is possible to choose compatible orientations for all the Q_i's.

Note that, in general, there can be a number of different templates associated with a given A. In particular, we were free to choose any subset of A_{ij} of the β_{ik}'s to attach to α_j. Also the α's and β's are naturally oriented as subsets of the plane, and we were free to choose whether the attaching maps from β_{ik} to α_j preserved or reversed this orientation.

On the other hand, it is clear that from the template $(T, \bar{\phi}_t)$ one can reconstruct the matrix A and hence the subshift σ_A. This justifies some of the arrows in our initial chart.

We now include some examples which will be important later (see Figure (2.3)). Note that there are three different abstract templates associated with the matrix (2) (depending on whether both, one or neither of the arms have a twist).

Matrix | Graph | Template

(2) or $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$

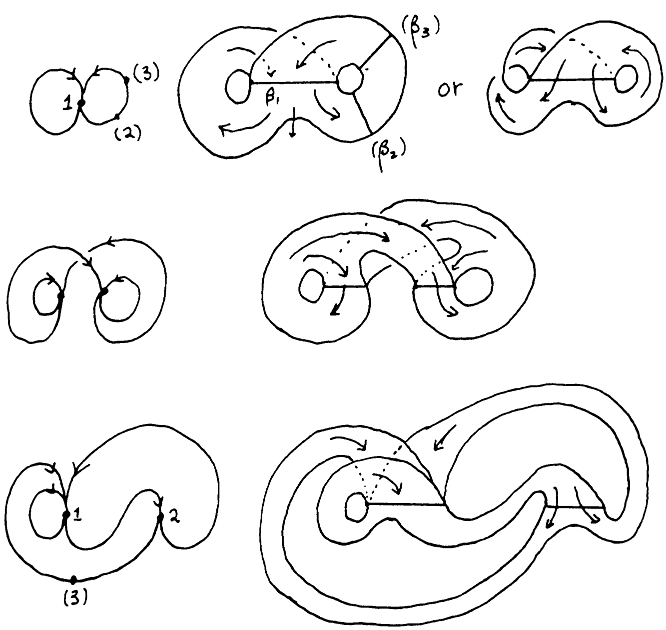


FIGURE (2.3)

(2.3) DEFINITION. A *Lorenz template* $(T, \bar{\phi}_t)$ is any of the three templates corresponding to the matrix (2).

Note, however, that there are infinitely many distinct (up to isotopy) ways of embedding each Lorenz template.

The point of investigating templates is that it is often possible to analyse (to some extent) the kinds of knots which occur as closed orbits of the semiflow $\bar{\phi}_t$ on T and techniques of [**B-W I, II**] enable one to show under certain hypotheses a correspondence between knotted periodic orbits of a smooth flow and knotted orbits on a template.

The first step in the proof of our main theorem is the following result which relies heavily on techniques of [**B-W II**]:

(2.4) PROPOSITION. *If ϕ_t is a smooth flow on $\mathbf{R}^3$ (or S^3) which possesses a hyperbolic closed orbit with a transversal homoclinic orbit, then there exists an embedded Lorenz template $(T, \bar{\phi}_t)$ in $\mathbf{R}^3$ (or S^3) such that every closed orbit of $\bar{\phi}_t$ is isotopic to a closed orbit of ϕ_t.*

PROOF. We first observe that the Smale-Birkhoff theorem (see [**G-H**, p. 252[2]]) asserts the existence of a transversal S to the flow whose return map ρ possesses a compact hyperbolic invariant set Λ. Moreover $\rho\colon \Lambda \to \Lambda$ is topologically conjugate to the subshift of finite type corresponding to the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

Now we apply techniques of [**B-W II** (see especially Theorem (2.1) and Lemma (2.3))].

Although our flow ϕ_t does not strictly satisfy the hypothesis of (2.1) of [**B-W II**] since we have hyperbolicity not on all of the chain recurrent set but only part of it (the orbits of points in Λ), we can nevertheless construct a template for this part exactly as in the proof of this theorem. (In [**B-W II**] templates are called knotholders.) The closed orbits of the resulting template will not be in one-to-one correspondence with all closed orbits of ϕ_t but only with the orbits containing a point of Λ. However, this will be adequate for our purposes.

The template $(T, \bar{\phi}_t)$ so constructed will correspond to the matrix A. This is easy to see from its construction. In Figure (2.4) we illustrate the template T so constructed (though this diagram is accurate only as an abstract template—no claim is made about the embedding). From this it is clear that there exists a Lorenz template $T' \subset T$ with semiflow the restriction to T' of $\bar{\phi}_t$. Clearly any closed orbit in

[2] In their notation, A is a $(k + l) \times (k + l)$ matrix and not $(k + l + 1) \times (k + l + 1)$ as stated.

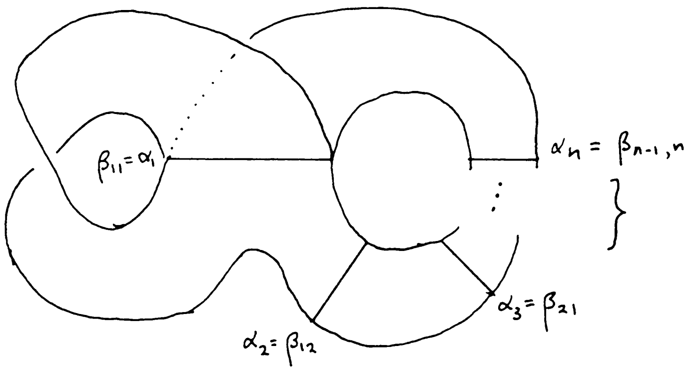


FIGURE (2.4)

T' is also in T (in fact the reverse containment is also true). Now $(T', \bar{\phi}_t)$ is the desired template. Q.E.D.

This result reduces the proof of our main theorem to showing that every Lorenz template (no matter how embedded) contains infinitely many different knot types among its closed orbits.

3. Alexander's trick. In this section we are concerned with taking an arbitrarily embedded template in $\mathbf{R}^3$ and arranging it, via isotopy, in such a way that it can be more easily analysed.

(3.1) DEFINITION. A template $H, \bar{\phi}_t$ in $\mathbf{R}^3$ is arranged as a *flat braid* provided there is a plane $E \subset \mathbf{R}^3$, an orthogonal projection $\pi: \mathbf{R}^3 \to E$, and a point $\mathcal{O} \in E$ such that

(a) the projection of any orbit of $\bar{\phi}_t$ proceeds clockwise around $\mathcal{O}$,

(b) $\pi|_H$ is an immersion.

If (b) is replaced by

(b′) $\pi|_H$ is an immersion except for a finite number of half-twists, which are isolated,

we say H is arranged as a *twisted braid.*

(3.2) PROPOSITION. *Any orientable template embedded in* $\mathbf{R}^3$ *can be arranged as a flat braid. Without assuming orientability a template can be so arranged, except for finitely many half-twists.*

PROOF. Fairly standard general position arguments allow us to arrange the template so that its projection onto a plane E will have an image consisting of nine types of pieces (see Figure (3.1))

In the oriented case, we first arrange that the branch set charts (type 3) are coherently oriented. That is, choosing a normal field to the template H, we arrange that the normal is always upward (say) from E on these charts. One can just turn any offending chart over. In either case, our next step is to arrange that the semiflow $\bar{\phi}_t$ on these charts is *clockwise* about $\mathcal{O}$. Next, we move all twists backward (against the flow) until they are directly below the various branch sets maintaining the

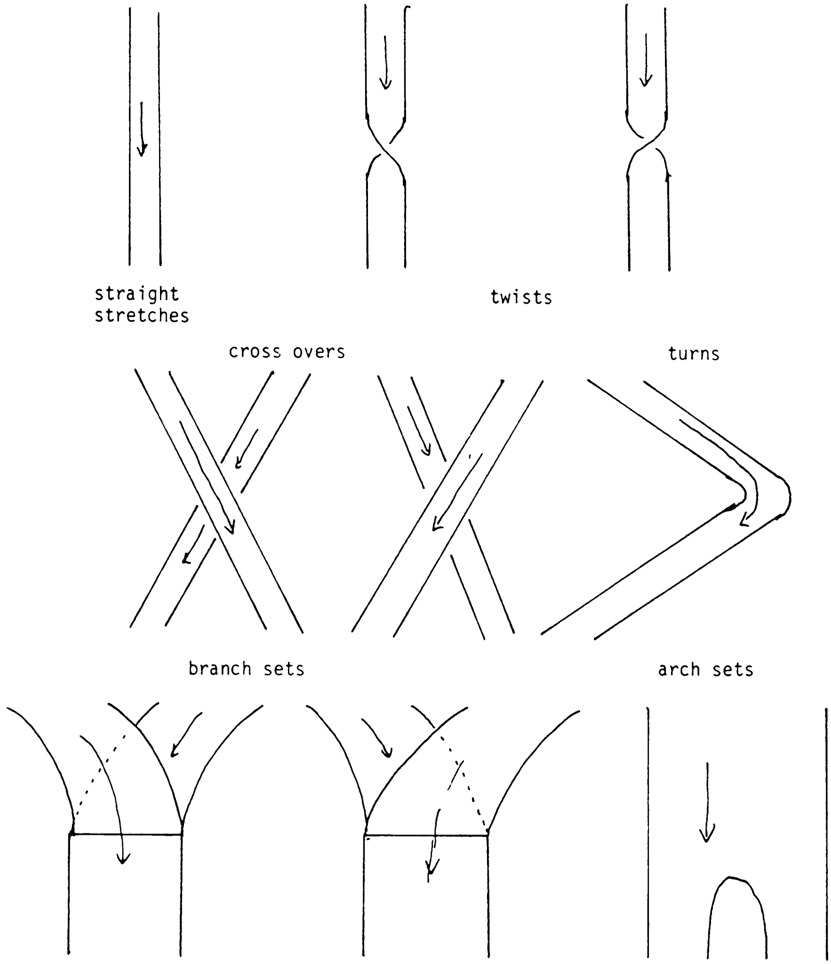


FIGURE (3.1)

clockwise behavior about $\mathcal{O}$ on these charts. Thus other than charts like Figure (3.2), our picture of H has (1) a finite number of turning points connected by straight sections and (2) a finite number of cross-overs, which occur only between turns. We can assume that none of the straight sections points directly toward or away from $\mathcal{O}$. Then some finite number, say m, of our straight sections are passing around $\mathcal{O}$ in the *wrong* (counterclockwise) direction. Alexander's trick is to show that certain simple

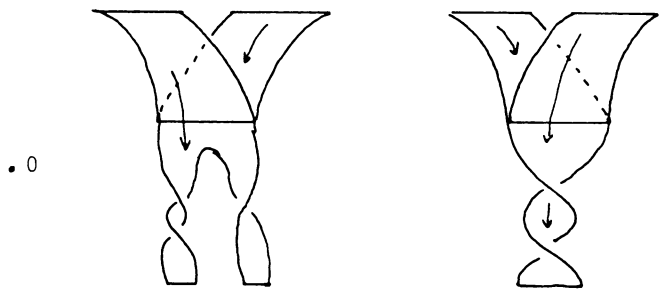


FIGURE (3.2)

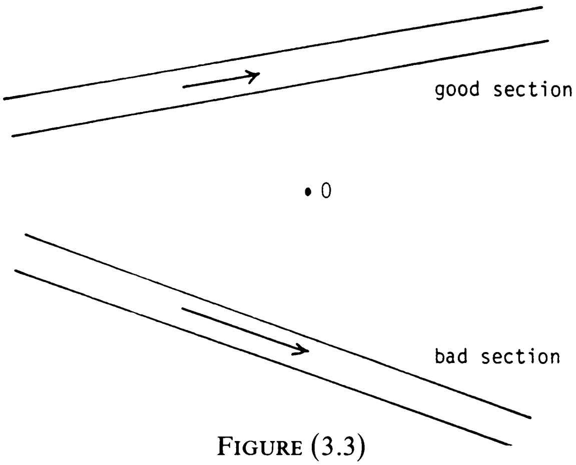


FIGURE (3.3)

isotopies reduce the number m by 1 and thus completes the proof by induction (see Figure (3.3)).

Thus let $L \subset H$ be an offending straight section between turns at, say, A and B. Renumber $A = A_0$, and locate $A_1, A_2, \ldots, A_n = B$ along the straight stretch (not at cross-overs), so that between A_i and A_{i+1}, all the cross-overs are of the same type. That is, either all go *under* L (A_1, A_2 in Figure (3.4)) or all go over L (A_0, A_1 in Figure (3.4)). Now consider a segment A_iA_{i+1} which passes under no part of H. Then it can be replaced with a V-shaped strip (vertically hatched in the figure) passing over all of H with both sections proceeding in the good direction around $\mathcal{O}$. Those segments A_j, A_{j+1} lying under some part of H, but over none, can likewise be displaced to a two segment strip, passing under H and going around $\mathcal{O}$ in the clockwise direction (horizontally hatched in our figure).

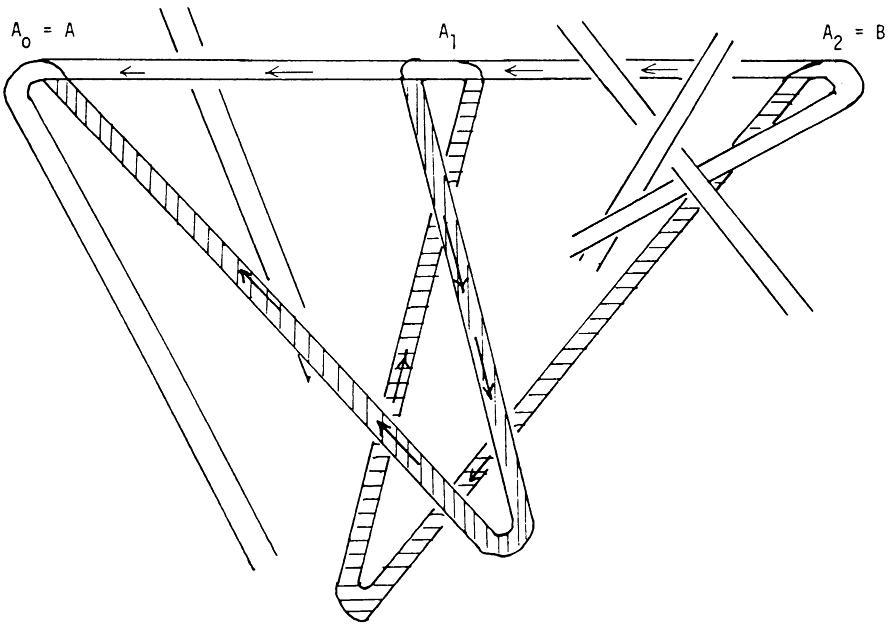


FIGURE (3.4)

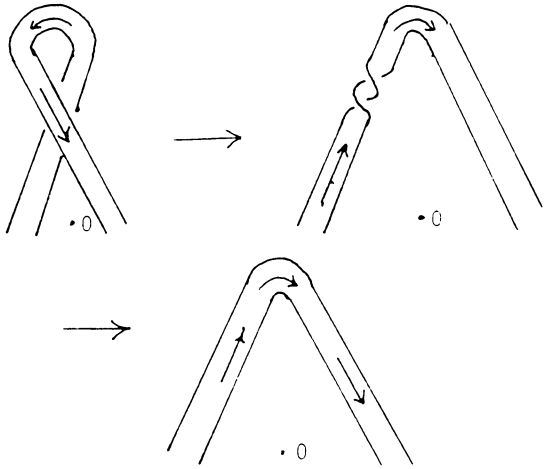


FIGURE (3.5)

We have, however, introduced two "curls" in our figure (near A_1 and A_2) and this happens quite generally (this twisting occurs because the strip has positive width—it did not arise in Alexander's work). These curls are easily replaced with full twists, which can in turn be moved over to a branch set (see Figure (3.5)).

Thus by induction we are finished, except that in the orientable case, we need to remove any twists that occur. Recall that these occur just beneath the branch sets. There must be an *even* number of half-twists, as otherwise, by following the semiflow around to the next branch set chart, we would arrive with downward directed normal. It is an easy matter to remove these in pairs, as in Figure (3.6). This completes the proof.

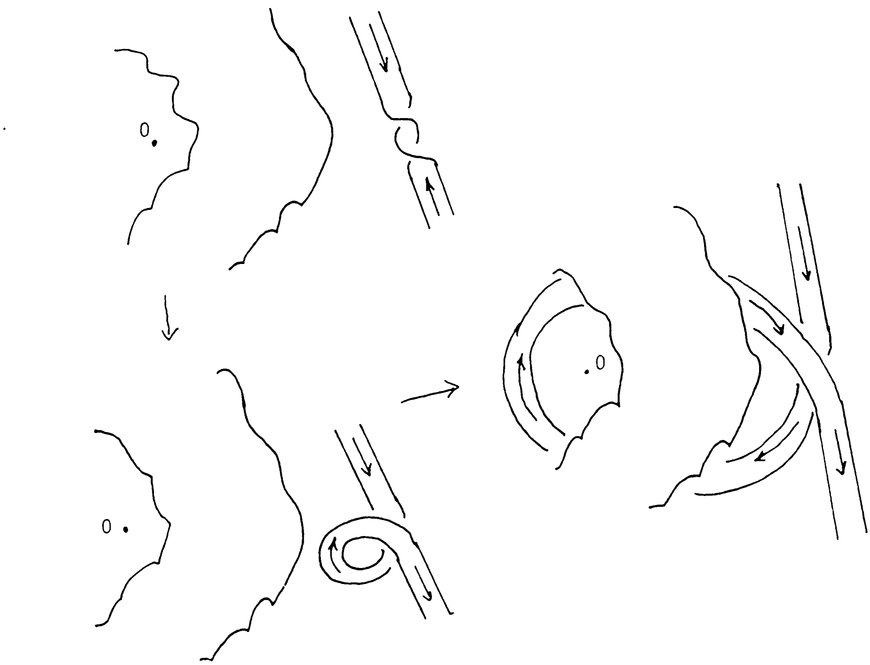


FIGURE (3.6)

4. Conclusion. We are now prepared to complete the proof of our main theorem. By our remarks in the introduction we can assume the transversal homoclinic point hypothesis. Thus by (2.4) we will be finished if we can prove the following

(4.1) PROPOSITION. *Any embedded Lorenz template contains infinitely many knot types among its closed orbits.*

The proof of this result will consist of several lemmas.

We first need the following easy facts, whose proofs can be found in §(2.4) of **[B-W I]**.

(4.2) LEMMA. *If the two "arms" of a Lorenz template $(T, \bar{\phi}_t)$ are labelled x and y, then there is a one-to-one correspondence between points on the branch line of T which lie on periodic orbits of $\bar{\phi}_t$ and finite aperiodic words in the indeterminates x and y. The correspondence is given by assigning to p the sequence of x's and y's corresponding to the arms that p traverses under the semiflow $\bar{\phi}_t$. Moreover, if the template T is oriented, the order in which these points lie on the branch line is the alphabetical order of their corresponding words (or its reverse).*

We want also to reduce the analysis of an arbitrary Lorenz template to an analysis of an oriented one.

(4.3) LEMMA. *Given an unorientable Lorenz template $(T, \bar{\phi}_t)$ embedded in $\mathbf{R}^3$ (or S^3), there exists an embedded orientable template $(T', \bar{\phi}'_t)$ having the property that, with finitely many exceptions, every closed orbit on T' is isotopic to a closed orbit on T.*

PROOF. If the arm labelled y is twisted in the template T, one can form a new template by cutting open along the closed orbit in T corresponding to the word "y", and enlarging the arch of T (see Figure (4.1)). If the arm labelled x was untwisted, the resulting template is orientable, and clearly there is a correspondence of closed orbits except those on the cut. If the x arm is also twisted we make another cut. Q.E.D.

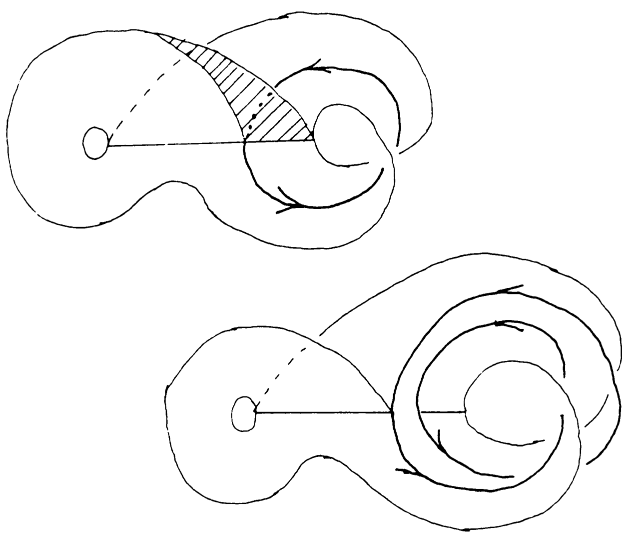


FIGURE (4.1)

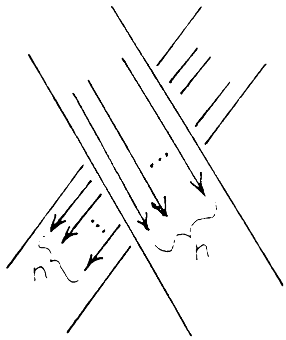


FIGURE (4.2)

(4.4) PROPOSITION. *For any Lorenz template, one of the following four families of words corresponds to a family of closed orbits containing infinitely many knot types*: $x^n y$, xy^n, $x^n y^n$, *and* $x(xy)^n$.

PROOF. We may assume our template is orientable by (4.3) and arranged as a flat braid by (3.2). Let α be the number of x-ribbon strands and β the number of y-ribbon strands in this braid. We can form the matrix of cross-overs of the x-ribbon and y ribbon, say

	x	y
x	a	b.
y	b	c

That is, a is the algebraic number of times the x-ribbon crosses itself, etc. Now in case $a \neq 0$, we use the first family and note that each x over x-ribbon crossing involves n^2 knot crossings. The x over y and y over y are, respectively, n and 1 crossings of the knot for each ribbon crossing. There remain the crossings at the branch set, but clearly there are at most n. Hence

$$2g(x^n y) > |an^2 + bn + c \pm n| - |\alpha n + \beta|$$

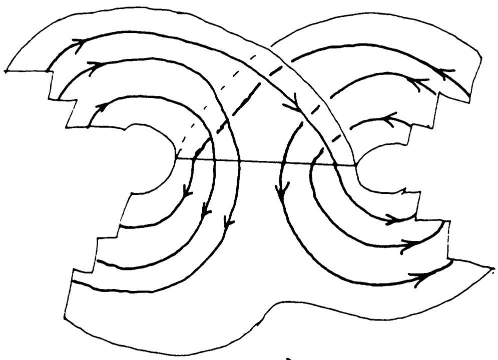
by Bennequin's theorem, as there are $\alpha n + \beta$ strands in $x^n y$ counted as a braid. As the right-hand side is quadratic in n, the genus takes on infinitely many values, and we are done. In case $c \neq 0$, we similarly can use the family xy^n. If $a = 0 = c$, but $b \neq 0$, then the family $x^n y^n$ has algebraically bn^2 crossings away from the branch line so that we get

$$2g(x^n y^n) \geqslant |bn^2 + B(n)| - |\alpha n + \beta|,$$

where $B(n)$ represents the crossings at the branch set. Hence it will suffice to show $B(n)$ is linear in n. We claim $B = \pm(2n - 1)$, as follows. The cyclic permutations of $x^n y^n$ occur in the following order:

$$x^n y^n < x^{n-1} y^n x < \cdots < xy^n x^{n-1}$$
$$< yx^n y^{n-1} < y^2 x^n y^{n-1} < \cdots < y^n x^n.$$

From this and (4.2) it is easy to see the situation is as in Figure (4.3).



(n = 3)

FIGURE (4.3)

The case $a = b = c = 0$ has actually been treated already, for as far as a lower bound on the genus is concerned, we could assume the template is the standard Lorenz attractor template. But for this, we know from **[B-W I]** that $x(xy)^n$ is the $n + 1$, n torus knot with $2g = n(n - 1)$. Hence $2g(x(xy)^n) \geqslant n^2 - n$. This completes the proof.

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